理论力学 A

2024 年秋季学期第四次习题课

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内容摘要

1. 知识点整理
   - 中心力与散射
   - Hamilton 力学
   - 刚体

中心力问题:

- 角动量守恒: $l = mr^2\dot{\theta} = \text{const.}$, 方向始终与 $\boldsymbol{r}\times\boldsymbol{v}$ 所在平面垂直;
- 径向运动与圆周运动稳定性分析:

$$E = \frac{1}{2}m\dot{r}^2 + V(r), \quad V(r) = U(r) + \frac{l^2}{2mr^2}, \quad \frac{\partial V}{\partial r} = 0, \quad \frac{\partial^2 V}{\partial r^2} > 0$$

- 轨道方程:
  1. 积分形式:

$$\theta = \pm\frac{l}{\sqrt{2m}}\int\frac{dr}{r^2\sqrt{E - V(r)}}, \quad \Delta\theta = \int_{r_1}^{r_2}\frac{l^2\,dr}{r^2\sqrt{2m(E-U) - l^2/r^2}}$$

  2. 一阶方程:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2 = \frac{2m}{l^2}(E - U), \quad u = \frac{1}{r}$$

3. 二阶方程 (Binet 公式)

$$\frac{d^2u}{d\theta^2}+u=-\frac{mF}{l^2u^2},\quad F=-\frac{\partial U}{\partial r}$$

- 中心力作用下轨道关于拱线对称，周期运动(轨道闭合)的条件：

$$\Delta\theta=q\pi,\quad q\in\mathbb{Q}$$

散射问题：

- 散射角：$\Theta=\pi-2\theta_b$ (排斥力), $\Theta=|2\theta_b-\pi-2n\pi|$ (吸引力),θ_b:

$$\theta_b=\int_{r_{\min}}^{\infty}\frac{bdr}{r^2\sqrt{1-U/E-b^2/r^2}}$$

- 微分散射截面：

$$\frac{d\sigma}{d\Omega}=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|\quad(\text{轴对称情形})$$

Legendre 变换与 Hamilton 方程

- 利用 Legendre 变换可从 Lagrange 函数给出 Hamilton 函数，需要满足 Hess 条件：

$$H(q,p,t) = p_k\dot{q}_k - L(q,\dot{q},t), \quad \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0$$

- Hamilton 方程：

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}}$$

或写成正则变量的形式：

$$\dot{\xi} = \Omega\frac{\partial H}{\partial \xi} = [\xi, H]_\xi, \quad \Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \quad \xi = (q,p)$$

- Ω 矩阵的性质：

$$\Omega^T = \Omega^{-1} = -\Omega, \quad \Omega_{\alpha\beta}\Omega_{\gamma\beta} = \delta_{\alpha\gamma}$$

Poission 括号

- Poission 括号:

$$[A,B]_\xi = \frac{\partial A}{\partial \xi_\alpha}\Omega_{\alpha\beta}\frac{\partial B}{\partial \beta} = \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k}$$

- 基本 Poission 括号:

$$[q_k, p_l] = \delta_{kl}, \quad [q_k, q_l] = [p_k, p_l] = 0$$

- Poission 括号的性质(Poission 代数):

$$[A,B] = -[B,A], \quad [A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$
$$[A,BC] = [A,B]C + B[A,C], \quad [AB,C] = A[B,C] + [A,C]B$$

- 力学量的运动方程及 Taylor 展开:

$$\dot{f} = \frac{\partial f}{\partial t} + [f,H]_\xi, \quad f(t+\tau) = \exp(-\tau D_H)f, \quad D_H f = [H,f]$$

- 判断 Hamilton 体系 $\Leftrightarrow [\dot{\xi}_\alpha, \xi_\beta] + [\xi_\alpha, \dot{\xi}_\beta] = 0$

受限正则变换的条件：

- 基本 Poission 括号的不变性：$\xi \mapsto \eta, [\eta_\alpha, \eta_\beta]_\xi = \Omega_{\alpha\beta}$
- 辛条件：

$$M_{\alpha\beta} = \frac{\partial \eta_\alpha}{\partial \xi_\beta}, \quad M^T \Omega M = \Omega$$

- 可积条件：

$$\boxed{p_k dq_k - H = P_i dQ_i - K + dF(q, p, t)}$$

$$\Rightarrow \frac{\partial F}{\partial q_k} = p_k - P_i \frac{\partial Q_i}{\partial q_k}$$

$$\frac{\partial F}{\partial p_k} = -P_i \frac{\partial Q_i}{\partial p_k}$$

$$K = H + P_i \frac{\partial Q_i}{\partial t} + \frac{\partial F}{\partial t}$$

正则变换的性质：

- hamilton 体系的演化可视为正则变换
- Liouville 体积定理：$\Gamma(t) = \Gamma(0)$
- Liouville 定理：

$$n = \frac{\Delta N}{\Delta \Gamma} = n_0, \quad \rho = \frac{n}{N} = n_0, \quad J_\alpha = \rho_\alpha \Omega_{\alpha\beta} \frac{\partial H}{\partial \beta},$$

$$\frac{\partial \rho}{\partial t} + \partial_\alpha J_\alpha = 0$$

正则变换的分类：

- 第一类正则变换：q, Q 独立，$F_1(q, Q, t) = F(q, p, t)$,
- 第二类正则变换：q, P 独立，$F_2(q, P, t) = F(q, p, t) + Q_k P_k$,

$$p_k = \frac{\partial F_2}{\partial q_k}, \quad Q_k = \frac{\partial F_2}{\partial P_k}, \quad K = H + \frac{\partial F_2}{\partial t}$$

- 第三类正则变换：p, Q 独立，$F_3(p, Q, t) = F(q, p, t) - q_k p_k$,
- 第四类正则变换：p, P 独立，$F_4(p, P, t) = F(q, p, t) - q_k p_k + Q_l P_l$.

Hamilton-Jacobi 理论：

- Hamilton-Jacobi 方程：

$$S(q, P, t) = F_2(q, P, t), \quad \boxed{-\frac{\partial S}{\partial t} = H(q, \frac{\partial S}{\partial q}, t)}$$

- 自洽体系：$H = H(q, p), \quad S = T(t) + W(q)$

$$-\frac{\partial T}{\partial t} = H(q, \frac{\partial W}{\partial q}), \quad H(q, \frac{\partial W}{\partial q}) = P_1, \quad \Rightarrow T = -P_1 t + C$$

若 H 不显含某个 q_k，则令对应的 p_k 为运动常数 P_2，有：

$$p_n = \frac{\partial S}{\partial q_n} = P_n = \frac{\partial W}{\partial q_n}, \Rightarrow W = P_n q_n$$

刚体运动学

- 欧拉运动学方程(只写了一种形式,考试如要用大概率会给):

$$\begin{cases} \omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases}$$

- 惯量张量相关:

$$\boldsymbol{L} = J\cdot\boldsymbol{\omega}, \quad J = \int dm(r^2\mathbf{1} - \mathbf{rr})$$

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L} = \frac{1}{2}\boldsymbol{\omega}\cdot J\cdot\boldsymbol{\omega} = \frac{1}{2}J_{ij}\omega_i\omega_j$$

- 正交轴定理:$J_{11} + J_{22} \geqslant J_{33}$,平行轴定理:$J = J^* + J_C$
- 主轴与主转动惯量:本质上是惯量张量矩阵本征值问题的解。

刚体动力学

- Euler 动力学方程：

$$\begin{cases} \tau_1 = J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 \\ \tau_2 = J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 \\ \tau_3 = J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 \end{cases}$$

- Euler 陀螺：$\boldsymbol{\tau} = 0$, L^2, T 守恒
- Lagrange 陀螺：

$$\begin{aligned} L &= \frac{1}{2}J_{12}(\omega_1^2 + \omega_2^2) + \frac{1}{2}J_3\omega_3^2 - mgl\cos\theta \\ &= \frac{1}{2}J_{12}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}J_3(\dot{\varphi}\cos\theta + \dot{\psi})^2 - mgl\cos\theta \end{aligned}$$

p_ψ, p_φ, E 守恒，其中：

$$E = \frac{1}{2}J_{12}\dot{\theta}^2 + \frac{(p_\varphi - p_\psi\cos\theta)^2}{2J_{12}\sin^2\theta} + \frac{p_\psi^2}{2J_3} + mgl\cos\theta$$