理论力学 A 第二次作业答案

助教 唐延宇

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Problem 1. 以下假设 $\vec{a}=a_i\vec{e}_i, \vec{b}=b_i\vec{e}_i, \vec{c}=c_i\vec{e}_i$, 其中 $\vec{e}_i$ 为直角坐标下的单位基矢量:

$$\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{a}\times(\vec{e}_i\epsilon_{ijk}b_jc_k) \\
&= \epsilon_{lmi}\vec{e}_l a_m \epsilon_{ijk} b_j c_k \\
&= \delta_{jl}\delta_{km}\vec{e}_l a_m b_j c_k - \delta_{jm}\delta_{kl}\vec{e}_l a_m b_j c_k \\
&= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}
\end{aligned}$$

$$\begin{aligned}
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\epsilon_{ijk}\vec{e}_i a_j b_k)\cdot(\epsilon_{lmn}\vec{e}_l c_m d_n) \\
&= \epsilon_{ijk}\epsilon_{imn} a_j b_k c_m d_n \\
&= \delta_{jm}\delta_{kn} a_j b_k c_m d_n - \delta_{jn}\delta_{km} a_j b_k c_m d_n \\
&= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})
\end{aligned}$$

$$\begin{aligned}
&\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b}) \\
&= (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}+(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}+(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b} \\
&= 0
\end{aligned}$$

Problem 2. (1) 根据: $C_i=\frac{1}{2}\epsilon_{ijk}T_{jk}$, 可知:

$$\begin{aligned}
\epsilon_{ijk}C_k &= \frac{1}{2}\epsilon_{ijk}\epsilon_{kmn}T_{mn} \\
&= \frac{1}{2}(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})T_{mn} \\
&= \frac{1}{2}(T_{ij}-T_{ji}) \\
&= T_{ij}
\end{aligned}$$

(2)

点乘	矢量	轴矢量
矢量	标量	赝标量
轴矢量	赝标量	标量

叉乘	矢量	轴矢量
矢量	赝矢量	矢量
轴矢量	矢量	赝矢量

Problem 3. (1) 根据转动公式的分量形式：

$$\begin{aligned} x_i' &= x_i \cos\theta + n_i n_j (1-\cos\theta) x_j + \epsilon_{ikj} n_k x_j \sin\theta \\ &= [\delta_{ij}\cos\theta + n_i n_j (1-\cos\theta) - \epsilon_{ijk} n_k \sin\theta] x_j \end{aligned}$$

又 $x_i' = R_{ij} x_j$，得：

$$R_{ij} = \delta_{ij}\cos\theta + n_i n_j (1-\cos\theta) - \epsilon_{ijk} n_k \sin\theta$$

据此可得：

$$R = I\cos\theta + \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{pmatrix} (1-\cos\theta) + \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \sin\theta$$

(2)

$$\hat{n} = \frac{1}{\sqrt{3}}(1,1,1), R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \left(R^T \text{亦可}\right)$$

(3)

$$\begin{aligned} R_{ik}R_{jk} &= [\delta_{ik}\cos\theta + n_i n_k (1-\cos\theta) - \epsilon_{ikl} n_l \sin\theta][\delta_{jk}\cos\theta + n_j n_k (1-\cos\theta) - \epsilon_{jkm} n_m \sin\theta] \\ &= \delta_{ik}\delta_{jk}\cos^2\theta + n_i n_j n_k n_k (1-\cos\theta)^2 + \epsilon_{ikl}\epsilon_{jkm} n_l n_m \sin^2\theta \\ &+ \delta_{ik} n_j n_k (1-\cos\theta)\cos\theta - \delta_{ik}\epsilon_{jkm} n_m \cos\theta\sin\theta - \epsilon_{ikl} n_j n_l n_k (1-\cos\theta)\sin\theta \\ &+ \delta_{jk} n_i n_k (1-\cos\theta)\cos\theta - \delta_{jk}\epsilon_{ikm} n_m \cos\theta\sin\theta - \epsilon_{jkm} n_m n_i n_k (1-\cos\theta)\sin\theta \end{aligned}$$

由于 $n_i n_j$ 为对称张量，ϵ_{ijk} 为全反对称张量，则根据上次作业的结论可知，形如 $\epsilon_{ijk} n_i n_j$ 的项事实上对求和没有贡献. 再虑及 $n_k n_k = \hat{n}^2 = 1$, 那么：

$$\begin{aligned} R_{ik}R_{jk} &= \delta_{ij}\cos^2\theta + n_i n_j (1-\cos\theta)^2 + (\delta_{ij}\delta_{ml} - \delta_{im}\delta_{jl}) n_m n_l \sin^2\theta \\ &+ 2 n_i n_j (1-\cos\theta)\cos\theta - (\epsilon_{ijm} + \epsilon_{jim}) n_m \cos\theta\sin\theta \\ &= \delta_{ij}(\cos^2\theta + \sin^2\theta) + n_i n_j [(1-\cos\theta)^2 - \sin^2\theta + 2(1-\cos\theta)\cos\theta] \\ &= \delta_{ij} \end{aligned}$$

Problem 4. 设 $\overleftrightarrow{T} = T_{ij}\hat{x}_i\hat{x}_j = T_{ij}'\hat{x}_i'\hat{x}_j'$, 其中 $\hat{x}_i, \hat{x}_i'$ 分别为原参考系与转动参考系中的单位矢量，由于在转动参考系中，转动参考系的直角坐标基矢自然是常矢量，有：

$$\left(\frac{\mathrm{d}\overleftrightarrow{T}}{\mathrm{d}t}\right)_{rot} = \left(\frac{\mathrm{d}}{\mathrm{d}t}(T_{ij}'\hat{x}_i'\hat{x}_j')\right)_{rot} = \left(\frac{\mathrm{d}T_{ij}'}{\mathrm{d}t}\hat{x}_i'\hat{x}_j'\right)_{rot} = \frac{\mathrm{d}T_{ij}'}{\mathrm{d}t}\hat{x}_i'\hat{x}_j'$$

而：

$$
\begin{aligned}
\frac{\mathrm{d}\overleftrightarrow{T}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(T'_{ij}\hat{x}'_i\hat{x}'_j) \\
&= \frac{\mathrm{d}T'_{ij}}{\mathrm{d}t}\hat{x}'_i\hat{x}'_j + T'_{ij}\frac{\mathrm{d}\hat{x}'_i}{\mathrm{d}t}\hat{x}'_j + T'_{ij}\hat{x}'_i\frac{\mathrm{d}\hat{x}'_j}{\mathrm{d}t} \\
&= \left(\frac{\mathrm{d}\overleftrightarrow{T}}{\mathrm{d}t}\right)_{rot} + T'_{ij}\vec{\omega}\times\hat{x}'_i\hat{x}'_j + T'_{ij}\hat{x}'_i\vec{\omega}\times\hat{x}'_j \\
&= \left(\frac{\mathrm{d}\overleftrightarrow{T}}{\mathrm{d}t}\right)_{rot} + T'_{ij}\vec{\omega}\times\hat{x}'_i\hat{x}'_j - T'_{ij}\hat{x}'_i\hat{x}'_j\times\vec{\omega} \\
&= \left(\frac{\mathrm{d}\overleftrightarrow{T}}{\mathrm{d}t}\right)_{rot} + \vec{\omega}\times\overleftrightarrow{T} - \overleftrightarrow{T}\times\vec{\omega}
\end{aligned}
$$

Problem 5. (1) 据柱坐标定义：

$$
\begin{cases}
x = s\cos\phi = \sqrt{\xi\eta}\cos\phi \\
y = s\sin\phi = \sqrt{\xi\eta}\sin\phi \\
z = z = \frac{1}{2}(\xi-\eta)
\end{cases}
$$

在 $x-z$ 平面内，有 $y=0$，不妨 $\phi=0$，即：

$$
\begin{cases}
x = \sqrt{\xi\eta} \\
z = \frac{1}{2}(\xi-\eta)
\end{cases}
$$

η 曲线（消去 η）:

$$z = \frac{1}{2}(\xi - \frac{x^2}{\xi})$$

ξ 曲线（消去 ξ）:

$$z = \frac{1}{2}(\frac{x^2}{\eta} - \eta)$$

显见曲线均为抛物线.

(2)

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \sqrt{\xi\eta}\cos\phi\hat{x} + \sqrt{\xi\eta}\sin\phi\hat{y} + \frac{1}{2}(\xi-\eta)\hat{z}$$

则有：

$$
\begin{aligned}
\left|\frac{\partial\vec{r}}{\partial\xi}\right|\hat{\xi} &= \frac{\partial\vec{r}}{\partial\xi} = \frac{1}{2}\sqrt{\frac{\eta}{\xi}}(\cos\phi\hat{x} + \sin\phi\hat{y}) + \frac{1}{2}\hat{z} \\
\left|\frac{\partial\vec{r}}{\partial\eta}\right|\hat{\eta} &= \frac{\partial\vec{r}}{\partial\eta} = \frac{1}{2}\sqrt{\frac{\xi}{\eta}}(\cos\phi\hat{x} + \sin\phi\hat{y}) - \frac{1}{2}\hat{z} \\
\left|\frac{\partial\vec{r}}{\partial\phi}\right|\hat{\phi} &= \frac{\partial\vec{r}}{\partial\phi} = \sqrt{\xi\eta}(-\sin\phi\hat{x} + \cos\phi\hat{y})
\end{aligned}
$$

可得：$\hat{\xi}\cdot\hat{\eta} = \hat{\xi}\cdot\hat{\phi} = 0$

(3) 由正交性可得：

$$
\begin{aligned}
v^2 &= \left|\frac{\partial \vec{r}}{\partial \xi}\right|^2 \dot{\xi}^2 + \left|\frac{\partial \vec{r}}{\partial \eta}\right|^2 \dot{\eta}^2 + \left|\frac{\partial \vec{r}}{\partial \phi}\right|^2 \dot{\phi}^2 \\
&= \frac{\xi+\eta}{4}\left(\frac{\dot{\xi}^2}{\xi}+\frac{\dot{\eta}^2}{\eta}\right)+\xi\eta\dot{\phi}^2
\end{aligned}
$$

Problem 6. (1) 类似上题，可得：

η 曲线（消去 η）：

$$
\frac{\xi^2}{(\xi^2-1)\sigma^2}+\frac{z^2}{\xi^2\sigma^2}=1
$$

可见曲线为椭圆.

ξ 曲线（消去 ξ）：

$$
\frac{\xi^2}{(1-\eta^2)\sigma^2}-\frac{z^2}{\eta^2\sigma^2}=-1
$$

可见曲线为双曲线.

(2)

$$
\vec{r}=s\hat{s}+z\hat{z}=\sigma\sqrt{(\xi^2-1)(1-\eta^2)}\hat{s}+\sigma\xi\eta\hat{z}
$$

注意到 $\hat{s}$ 仅明显地依赖于 ϕ，$\hat{z}$ 是常量，有：

$$
\frac{\partial \hat{s}}{\partial \phi}=\dot{\hat{s}}/\dot{\phi}=\frac{\vec{\omega}\times\hat{s}}{\dot{\phi}}=\hat{\phi}
$$

式中 $\vec{\omega}=\frac{\mathrm{d}\phi}{\mathrm{d}t}\hat{z}$，所以：

$$
\begin{aligned}
\left|\frac{\partial \vec{r}}{\partial \xi}\right|\hat{\xi} &= \frac{\partial \vec{r}}{\partial \xi}=\sigma\xi\sqrt{\frac{1-\eta^2}{\xi^2-1}}\hat{s}+\sigma\eta\hat{z} \\
\left|\frac{\partial \vec{r}}{\partial \eta}\right|\hat{\eta} &= \frac{\partial \vec{r}}{\partial \xi}=-\sigma\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\hat{s}+\sigma\xi\hat{z} \\
\left|\frac{\partial \vec{r}}{\partial \phi}\right|\hat{\phi} &= \frac{\partial \vec{r}}{\partial \phi}=\sigma\sqrt{(\xi^2-1)(1-\eta^2)}\hat{\phi}
\end{aligned}
$$

可得：$\hat{\xi}\cdot\hat{\eta}=\hat{\xi}\cdot\hat{\phi}=0$

(3) 类似地，有：

$$
\begin{aligned}
v^2 &= \left|\frac{\partial \vec{r}}{\partial \xi}\right|^2 \dot{\xi}^2 + \left|\frac{\partial \vec{r}}{\partial \eta}\right|^2 \dot{\eta}^2 + \left|\frac{\partial \vec{r}}{\partial \phi}\right|^2 \dot{\phi}^2 \\
&= \sigma^2\left[(\xi^2-\eta^2)\left(\frac{\dot{\xi}^2}{\xi^2-1}+\frac{\dot{\eta}^2}{\eta^2-1}\right)+(\xi^2-1)(1-\eta^2)\dot{\phi}^2\right]
\end{aligned}
$$