理论力学 A 第十五次作业答案

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Problem 1.

如图所示，将角速度 ω 在 x_1, y_1 方向分解得：

$$\omega_{x_1} = \frac{a}{\sqrt{a^2+b^2}}\omega, \quad \omega_{y_1} = \frac{b}{\sqrt{a^2+b^2}}\omega$$

因此，x_1, y_1 方向的角动量分别为：

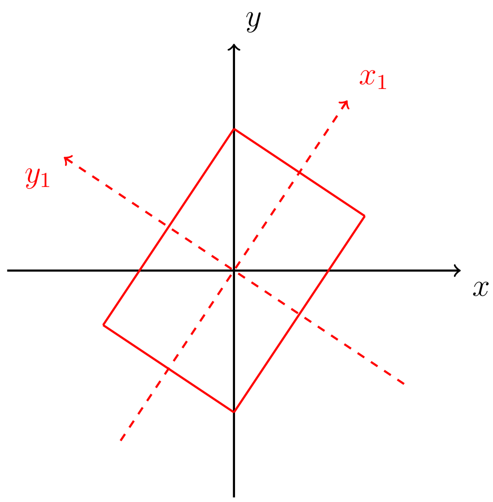
$$L_{x_1} = \frac{1}{12}\frac{ab^2}{\sqrt{a^2+b^2}}m\omega, \quad L_{y_1} = \frac{1}{12}\frac{a^2b}{\sqrt{a^2+b^2}}m\omega$$

在 x 方向进行合成，x 方向的角动量为：

$$L_x = \frac{1}{12}mab\omega\frac{b^2-a^2}{a^2+b^2}$$

相应地，力矩的大小为：

$$\tau = |\omega L_x| = \frac{1}{12}mab\omega^2\frac{b^2-a^2}{a^2+b^2}$$



Problem 2.

刚体的动力学方程为:

$$
\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 = -J_1\omega_2\omega_3 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 = J_2\omega_3\omega_1 \\
J_3\dot{\omega}_3 &= (J_1 + J_2)\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2
\end{aligned}
$$

初始条件:

$$
\omega_{10} = \Omega\cos\alpha, \quad \omega_{20} = 0, \quad \omega_{30} = \Omega\sin\alpha
$$

然而,上述微分方程组仍不易求解。我们利用角动量和能量守恒写出两个初次积分:

$$
\begin{aligned}
J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2 &= J_1\Omega^2\cos^2\alpha + J_3\Omega^2\sin^2\alpha = L^2 \\
J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2 &= J_1\Omega^2\cos^2\alpha + J_3\Omega^2\sin^2\alpha = 2E
\end{aligned}
$$

解得:

$$
\begin{aligned}
\omega_3^2 &= \frac{L^2 - 2J_1E + (J_2J_1 - J_2^2)\omega_2^2}{J_3^2 - J_1J_3} \\
\omega_1^2 &= \frac{L^2 - 2J_3E + (J_3J_2 - J_2^2)\omega_2^2}{J_1^2 - J_1J_3}
\end{aligned}
$$

结合 $J_1 = J_2\cos 2\alpha$, $J_1 + J_2 = J_3$ 代入得:

$$
\omega_3^2 = \Omega^2\sin^2\alpha - \omega_2^2\tan^2\alpha, \quad \omega_1^2 = \Omega^2\cos^2\alpha - \omega_2^2
$$

代入动力学方程得:

$$
\dot{\omega}_2 = \tan\alpha(\Omega^2\cos^2\alpha - \omega_2^2)
$$

结合 $\omega_{20} = 0$ 解得:

$$
\omega_2 = \Omega\cos\alpha\tanh(\Omega t\sin\alpha)
$$

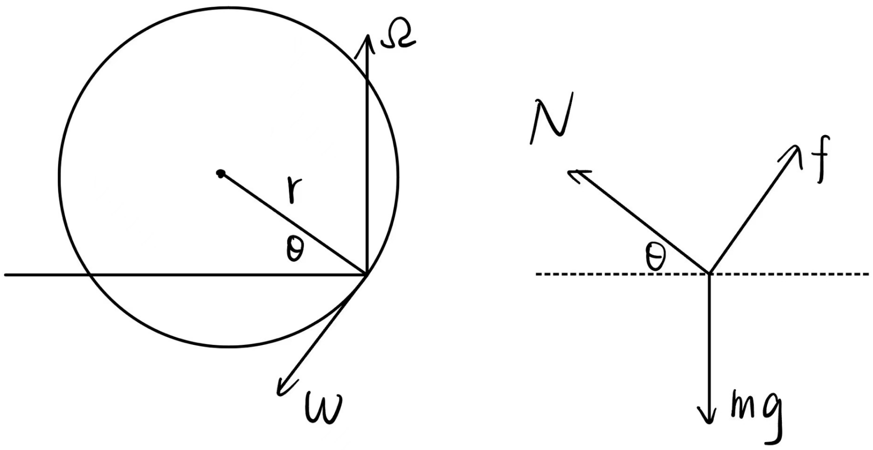
Problem 3.

设球的角速度为 ω, 则:

$$
\omega r = \Omega R
$$

垂直于转轴的角动量:

$$
L = \frac{7}{5}mr^2\omega
$$



并且:

$$\Omega L = NR - mgR$$

解得:

$$N = mg + \frac{7}{5}mr^2\Omega$$

Problem 4.

如图所示,球壳的角速度包含切点形成的大圆的转角速度 ω 和球心的进动角速度 Ω 两部分,ω 和 Ω 的关系可以表示为:

$$(R - r\cos\theta)\Omega = r(\omega - \Omega\cos\theta)$$

相对于质心的转动方程为:

$$\frac{2}{3}mr^2\omega\Omega\sin\theta = fr$$

质心的平动方程为:

$$f\cos\theta + N\sin\theta = mg$$

$$-f\sin\theta + N\cos\theta = m(R - r\cos\theta)\Omega^2$$

联立以上若干式解得:

$$\Omega = \sqrt{\frac{3g}{5R\tan\theta - 3r\sin\theta}}$$

Problem 5.

(1) 设刚体的进动角速度为 Ω, 自转角速度为 ω, 刚体质量为 m, 则刚体对原点的角动量沿杆方向的分量为:

$$L_1 = \frac{1}{2}mR^2(\omega + \Omega\cos\theta)$$

刚体对原点的角动量垂直于杆方向的分量为:

$$L_2 = \frac{1}{4}m(R^2 + 4l^2)\Omega\sin\theta$$

因此角动量垂直于转轴的分量为:

$$L_x = L_1\sin\theta - L_2\cos\theta = \frac{1}{2}mR^2\omega\sin\theta + \frac{1}{4}m(R^2 - 4l^2)\Omega\cos\theta\sin\theta$$

L_x 以水平向右为正方向, 刚体做规则进动时:

$$\Omega L_x = mgl\sin\theta$$

因此:

$$\frac{1}{2}R^2\Omega\omega + \frac{1}{4}(R^2 - 4l^2)\Omega^2\cos\theta = gl$$

在题中所给的情形下, 当 $\omega = 0$, 此时:

$$\Omega = \sqrt{\frac{8gl}{R^2 - 4l^2}}$$

需要满足:

$$R > 2L$$

(2) 当 $\Omega = \omega$ 时, 代入解得:

$$\Omega = \sqrt{\frac{8gl}{5R^2 - 4l^2}}$$

需要满足:

$$R > \frac{2}{\sqrt{5}}l$$

容易验证, $\Omega = -\omega$ 时, 不能满足条件。

(3) 约束力提供了刚体质心运动的向心力, 因此 Ω 越大约束力越大, 第一种情况约束力较大。

Problem 6.

(1) 欧拉运动学方程为:

$$\begin{cases}\omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}\end{cases}$$

Lagrange 陀螺的 Lagrange 量为

$$\begin{aligned}L &= \frac{1}{2}J_{12}(\omega_1^2+\omega_2^2) + \frac{1}{2}J_3\omega_3^2 - mgl\cos\theta \\ &= \frac{1}{2}J_{12}(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta) + \frac{1}{2}J_3(\dot{\varphi}\cos\theta+\dot{\psi})^2 - mgl\cos\theta\end{aligned}$$

首先得到两个守恒量 p_φ 和 p_ψ:

$$\begin{aligned}p_\psi &= \frac{\partial L}{\partial\dot{\psi}} = J_3(\dot{\varphi}\cos\theta+\dot{\psi}) \\ p_\varphi &= \frac{\partial L}{\partial\dot{\varphi}} = J_{12}\dot{\varphi}\sin^2\theta + p_\psi\cos\theta\end{aligned}$$

总能量:

$$\begin{aligned}E &= \frac{1}{2}J_{12}\dot{\theta}^2 + \frac{1}{2}J_{12}\dot{\varphi}^2\sin^2\theta + \frac{1}{2}J_3(\dot{\varphi}\cos\theta+\dot{\psi})^2 + mgl\cos\theta \\ &= \frac{1}{2}J_{12}\dot{\theta}^2 + \frac{(p_\varphi - p_\psi\cos\theta)^2}{2J_{12}\sin^2\theta} + \frac{p_\psi^2}{2J_3} + mgl\cos\theta\end{aligned}$$

等效势能为:

$$\begin{aligned}V_{\text{eff}}(\theta) &= E + \frac{1}{2}\left(\frac{1}{J_{12}} - \frac{1}{J_3}\right)p_\psi^2 - \frac{1}{2}J_{12}\dot{\theta}^2 \\ &= \frac{p_\varphi^2 + p_\psi^2 - 2p_\varphi p_\psi\cos\theta}{2J_{12}\sin^2\theta} + mgl\cos\theta\end{aligned}$$

在本题中,初始条件为:

$$\theta = 60^\circ, \quad \dot{\theta} = 0, \quad \dot{\varphi} = 2\sqrt{\frac{mgl}{3J_1}}, \quad \dot{\psi} = (3J_1 - J_3)\sqrt{\frac{mgl}{3J_1J_3^2}}$$

代入具体数值进行计算:

$$\begin{aligned}p_\psi &= J_3(\dot{\varphi}\cos\theta+\dot{\psi}) = \sqrt{3J_1mgl} \\ p_\varphi &= J_{12}\dot{\varphi}\sin^2\theta + p_\psi\cos\theta = \sqrt{3J_1mgl} \\ E &= \frac{1}{2}J_{12}\dot{\theta}^2 + \frac{(p_\varphi - p_\psi\cos\theta)^2}{2J_{12}\sin^2\theta} + \frac{p_\psi^2}{2J_3} + mgl\cos\theta = mgl + \frac{3J_1mgl}{2J_3}\end{aligned}$$

有效势能为：

$$\begin{aligned} V(\theta) &= \frac{p_\varphi^2 + p_\psi^2 - 2p_\varphi p_\psi \cos\theta}{2J_{12}\sin^2\theta} + mgl\cos\theta \\ &= mgl\left(\cos\theta + \frac{3(1-\cos\theta)}{\sin^2\theta}\right) \\ K &= \frac{1}{2}J_1\dot{\theta}^2 + V(\theta) = \frac{5}{2}mgl \end{aligned}$$

(2) 能量守恒方程可化为：

$$\frac{1}{2}J_1\dot{\theta}^2 + mgl\left(\cos\theta + \frac{3(1-\cos\theta)}{\sin^2\theta}\right) = \frac{5}{2}mgl$$

化简可得：

$$\dot{u}^2 = \frac{mgl}{J_1}(1-u)^2(2u-1), \quad u = \cos\theta$$

分离变量并积分，利用题中给出的积分公式即可解得：

$$\sec\theta = 1 + \operatorname{sech}\left(\sqrt{\frac{mgl}{J_1}}t\right)$$